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# The zero mass limit in Yang–Mills theory I

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**Abstract.** The zero mass limit of massive Yang–Mills theory is investigated and it is shown that there is a conflict between Lorentz invariance and the internal symmetry group in the theory. A necessary but not sufficient condition for the resolution of this conflict is the introduction of zero mass scalar fields.

It was pointed out some years ago (van Dam and Veltman 1970) that massive gravitational or Yang–Mills field theory has the unpleasant property of not tending to the conventional massless theory for arbitrarily small mass. In particular, by measuring the perihelion movement of Mercury or the bending of light by the sun, it is claimed (Iwasaki 1970, Zakharov 1970, van Dam and Veltman 1970) that a massive graviton is inconsistent with the data, however small the mass.

This paper is not about gravitational fields which may well have special characteristics. The Yang–Mills field, however, as the simplest example of a non-Abelian gauge field theory is of major interest; such non-Abelian gauge theories are now used as the underlying asymptotically-free theory of strong interactions. It is therefore disquieting to be told that although the essence of asymptotically-free theories is that mass terms can be neglected in some scaling limit, the mass $m \to 0$ limit of these theories is not the same as the underlying $m = 0$ theory. Indeed, as Sakurai (1960) emphasized very forcibly in his classic paper on the symmetries of the strong interactions, the internal symmetries of the interacting system should be taken as responsible for the dynamics; that is to say isotopic spin conservation, for example, implies the existence of the (massive) Yang–Mills field which we now identify with the rho-meson. It was these ideas which Gell-Mann (1961) generalized to SU(3) and it was essential that an octet of massive vector mesons should exist in a partially gauge-invariant theory which gave rise to the underlying fully gauge-invariant theory, constructed by analogy with the Yang–Mills theory, when these mass terms were turned off (Glashow and Gell-Mann 1961). So the fundamental modern ideas on the symmetry properties of the strong interactions are based on the existence of a smooth zero mass limit in a non-Abelian gauge theory, whereas van Dam and Veltman (1970) show that this limit does not exist for precisely this class of theories.

This is the problem that we investigate in this paper. We shall show that there is an underlying zero mass theory for which the $m \to 0$ limit is smooth, but that theory must contain zero mass scalar fields as well as vector fields. Indeed, we regard these scalar fields as being an integral part of the Yang–Mills field.

The discussion of the zero mass limit of vector meson theories naturally involves the ratio $E/m$ where $E$ is the energy of the particle; thus singularities at $m = 0$ correspond to the high energy behaviour of the theory and therefore involve the renormalizability of the theory. We shall not pursue these questions here. They will be treated in another paper (Dombey and Vayonakis 1976) where the self-interaction of the Yang–Mills field is investigated from the point of view developed in this paper. In particular, it is hoped to illustrate the connection between symmetry-breaking and renormalization which so often appears to be magical.

We begin by considering the zero mass limit of massive electrodynamics. It is now well known that although a photon mass term $(-\frac{1}{2}m^2A_\mu A_\mu)$ will break the gauge invariance of the Lagrangian, current conservation allows the massive theory to exist and the zero mass limit to be smooth as far as any physical observable is concerned. The problems of taking the zero mass limit of a massive vector meson theory arise from singularities at $m=0$ in $(a)$ the propagator which is of the form

$$\frac{\delta_{\mu\nu}+k_\mu k_\nu/m^2}{k^2+m^2} \tag{1}$$

for a virtual meson of momentum $k$, and $(b)$ the longitudinal polarization vector $\epsilon_{\rm L}$. This satisfies the constraint $k\cdot\epsilon_{\rm L}=0$ and so

$$\epsilon_{{\rm L}\mu}=(0,0,k_0/m,\mathrm{i}k_z/m) \tag{2}$$

where

$$k_\mu=(0,0,k_z,\mathrm{i}k_0) \tag{3}$$

with $k_z>0$.

A nice discussion of how current conservation allows a smooth zero mass limit in such Abelian theories has been given by Bell (1973). For our purposes here we will restrict the discussion to the difficulties caused in the zero mass limit by the longitudinal polarization vector $\epsilon_{\rm L}$.

$\epsilon_{\rm L}$ can be written

$$\epsilon_{{\rm L}\mu}=(m/k_0)\,e_{z\mu}+(k_z/mk_0)k_\mu \tag{4}$$

where

$$e_{z\mu}=(0,0,1,0). \tag{5}$$

So in a process in which there is one single photon, current conservation will cause the longitudinal mode to vanish as $m\to 0$, the amplitude of the decoupling being proportional to the Stueckelberg factor $m/k_0$†. In the limit at $m=0$ we have gauge invariance under the transformation $\epsilon\to\epsilon+\alpha k$, $k\cdot\epsilon=0$, $\alpha$ arbitrary, which is another way of saying that no longitudinal modes are present (Dombey 1964).

A covariant way of writing (4) is

$$\epsilon_{{\rm L}\mu}=k_\mu/m+(m/w)n_\mu \tag{6}$$

where $n_\mu$ is the lightlike vector

$$n_\mu=(0\quad 0\quad 1\quad -\mathrm{i}) \tag{7}$$

i.e. it is the lightlike vector conjugate to the lightlike vector given by the $m=0$ limit of $k$, and

$$w=k\cdot n. \tag{8}$$

† This is not strictly true (see Dombey 1964).

Now consider a problem with two external massive photons, for example, Compton scattering. Then the amplitude can be written

$$T=\epsilon_{\mu}^{(1)}T_{\mu\nu}\epsilon_{\nu}^{(2)} \tag{9}$$

where the indices (1) and (2) refer to the initial and final photon states. The amplitude for the scattering of longitudinal photons is thus

$$T_{\mathrm{LL}}=\epsilon_{\mathrm{L}\mu}^{(1)}T_{\mu\nu}\epsilon_{\mathrm{L}\nu}^{(2)}. \tag{10}$$

Using the decomposition (6) we obtain the result at $m=0$

$$T_{\mathrm{LL}}=\lim_{m\to 0}\,(k_{1\mu}T_{\mu\nu}k_{2\nu}/m^2)+k_{1\mu}T_{\mu\nu}n_{2\nu}/w_2+n_{1\mu}T_{\mu\nu}k_{2\nu}/w_1. \tag{11}$$

This is our main result. Clearly in a theory with

$$k_{1\mu}T_{\mu\nu}=T_{\mu\nu}k_{2\nu}=0 \tag{12}$$

then $T_{\mathrm{LL}}=0$ at $m=0$. In massive electrodynamics or any Abelian gauge theory this is the case. Equation (12), however, is not satisfied in a non-Abelian gauge theory. We now illustrate this point.

Take the triplet of fields $(\pi^+, \pi^0, \pi^-)$ to describe any isovector scalar or pseudoscalar particle of arbitrary mass and consider Compton scattering with a massive photon of mass $m$ off one of the charged components $\pi^+$ (say). Then to lowest order in $e^2$ (figure 1) the amplitude

$$T_{\mu\nu}=e^2[-(2p_1+k_1)_\mu(2p_2+k_2)_\nu/(2p_1\,.\,k_1-m^2)$$
$$+(2p_2-k_1)_\mu(2p_1-k_2)_\nu/(2p_1\,.\,k_2+m^2)+2\delta_{\mu\nu}]. \tag{13}$$

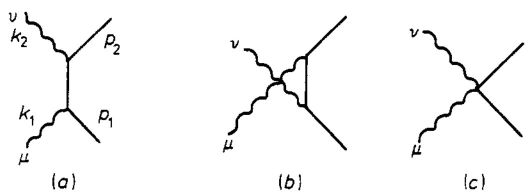


**Figure 1.**

By inspection

$$k_{1\mu}T_{\mu\nu}=T_{\mu\nu}k_{2\nu}=0$$

and so to this order the longitudinal modes will vanish in the $m\to 0$ limit. It is not difficult to show that this result holds in all orders of perturbation theory.

Now consider a Yang–Mills triplet $(\gamma^+, \gamma^0, \gamma^-)$ of mass $m$ described by the isotopic vector field $\boldsymbol{A}_\mu$ including the gauge-breaking mass term $(-\frac{1}{2}m^2\boldsymbol{A}_\mu\,.\,\boldsymbol{A}_\mu)$ in the Lagrangian. The lowest order diagrams for $\gamma^-\pi^+$ scattering are shown in figure 2. Hence in this case

$$T_{\mu\nu}=e^2\{-(2p_1+k_1)_\mu(2p_2+k_2)_\nu/(2p_1\,.\,k_1-m^2)$$
$$-[(p_1+p_2)\,.\,(k_1+k_2)\,\delta_{\mu\nu}-(2k_2-k_1)_\mu(p_1+p_2)_\nu$$
$$-(p_1+p_2)_\mu(2k_1-k_2)_\nu]/(2k_1\,.\,k_2+m^2)+\delta_{\mu\nu}\}. \tag{14}$$

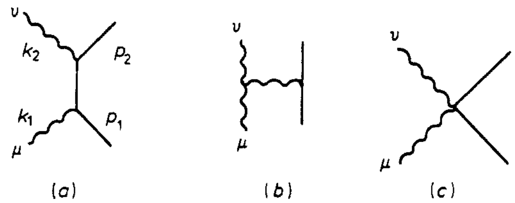


**Figure 2.**

Now

$$k_{1\mu}T_{\mu\nu}=Ak_{2\nu}, \qquad T_{\mu\nu}k_{2\nu}=Ak_{1\mu} \tag{15}$$

where

$$A=-\tfrac{1}{2}e^2(p_1+p_2)\cdot(k_1+k_2)/(2k_1\cdot k_2+m^2). \tag{16}$$

Note that $A$ contains the $t$ channel photon pole at $t=m^2$ where $t=-(k_1-k_2)^2=2k_1\cdot k_2+2m^2$. This arises from figure 2($b$) which involves the non-Abelian vertex connecting three Yang–Mills fields. It might be thought that conventional gauge invariance still held at $m=0$ even when equations (15) were satisfied as

$$(\epsilon_\mu^{(1)}+\alpha_1 k_\mu^{(1)})T_{\mu\nu}(\epsilon_\nu^{(2)}+\alpha_2 k_\nu^{(2)})=\epsilon_\mu^{(1)}T_{\mu\nu}\epsilon_\nu^{(2)} \tag{17}$$

for arbitrary (finite) $\alpha_1$, $\alpha_2$ at $m=0$. But the theorem (11) gives in this case at $m=0$

$$T_{\mathrm{LL}}=A\neq 0. \tag{18}$$

Hence the massive theory does not reduce correctly to the zero mass theory where, by Lorentz invariance, the massless vector meson can only have two transverse modes.

That the longitudinal mode cannot be transformed away in Yang–Mills theory by a gauge transformation should not be surprising; after all the gauge transformation valid at $m=0$

$$\boldsymbol{A}_\mu=\boldsymbol{A}_\mu+\partial_\mu\boldsymbol{\Lambda}+e\boldsymbol{\Lambda}\times\boldsymbol{A}_\mu \tag{19}$$

which keeps invariant the Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}_{\mu\nu} \tag{20}$$

where

$$\boldsymbol{G}_{\mu\nu}=\partial_\mu\boldsymbol{A}_\nu-\partial_\nu\boldsymbol{A}_\mu+e\boldsymbol{A}_\mu\times\boldsymbol{A}_\nu \tag{21}$$

is a different gauge transformation from the gauge transformation $A_\mu\to A_\mu+\partial_\mu\chi$ of classical electrodynamics. Although it is possible when considering a single charged photon state to choose $\boldsymbol{\Lambda}$ such that $\boldsymbol{\Lambda}\times\boldsymbol{A}_\mu=0$ and thus transform away the longitudinal component, it is not possible in general to choose non-commuting gauge fields $\boldsymbol{\Lambda}$ for more than one such photon. Although this result is not well known, it is not new; that a charged particle of zero mass and spin $J\geq 1$ could not be in general restricted to transverse modes was pointed out by Case and Gasiorowicz (1962) many years ago.

Thus we have rederived their result which shows that there is a conflict between Lorentz invariance which requires that zero mass vector particles can only have two transverse polarization states, and the internal non-Abelian symmetry which cannot transform away longitudinal modes. It is this conflict which is responsible for many difficulties encountered in Yang–Mills theory, not only that of the zero mass limit discussed here. For example, even if all external massless Yang–Mills fields are taken to have only the two transverse polarization states allowed by Lorentz invariance, in internal lines of loops they will have longitudinal modes which will not vanish when their internal momentum $q$ satisfies $q^2 = 0$. Thus the unitarity of the theory is violated unless physical scalar particles are introduced into loops to subtract out these unwanted longitudinal modes: these are the Fadeev–Popov ghosts.

In linearized gravitational theory, it has been pointed out that the zero mass limit of the massive theory would correspond to a Brans–Dicke theory, rather than the Einstein theory (van Nieuwenhuizen 1973). That is to say, if the appropriate Brans–Dicke theory is chosen as the underlying zero mass theory, a massive graviton can then be introduced into the theory smoothly. This procedure can be adopted with Yang–Mills theory too; it implies that there should be an underlying three component massless field which would consist of a massless two component vector field $\boldsymbol{A}_\mu$ and a massless scalar field $\boldsymbol{\phi}$. The basic Lagrangian is then

$$\mathscr{L} = -\tfrac{1}{4}\boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}_{\mu\nu} - \tfrac{1}{2}(\partial_\mu f\boldsymbol{\phi} + f\boldsymbol{\phi} \times \boldsymbol{A}_\mu)^2 + \bar{\mathscr{L}} \tag{22}$$

where $\bar{\mathscr{L}}$ involves extra terms necessary for renormalizability (Dombey and Vayonakis 1976) which are not relevant here.

The scalar fields $\boldsymbol{\phi}$ are introduced into the theory to describe the scattering of the longitudinal modes of the Yang–Mills field in a Lorentz invariant way. They thus describe the same triplet $(\gamma^+, \gamma^0, \gamma^-)$ as does the triplet of vector fields $\boldsymbol{A}_\mu$. The field $\boldsymbol{A}_\mu$ couples to the $\boldsymbol{\phi}$ current with coupling constant $f$. By comparison with equations (16) and (18) for the scattering amplitude $T_{\rm LL}$ at $m = 0$ we see that

$$f = \tfrac{1}{2}e \tag{23}$$

as $\boldsymbol{\phi}$ is taken to have no direct couplings with the matter field $\boldsymbol{\pi}$.

When the Yang–Mills field acquires mass the Lagrangian becomes

$$\mathscr{L} = -\tfrac{1}{4}\boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}_{\mu\nu} - \tfrac{1}{2}(\partial_\mu \boldsymbol{\phi} + f\boldsymbol{\phi} \times \boldsymbol{A}_\mu - m\boldsymbol{A}_\mu)^2 + \bar{\mathscr{L}}. \tag{24}$$

In terms of (24) we can reconsider our example of $\gamma^-\pi^+$ scattering. Now

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + Am^2 k_{1\mu}k_{2\nu}/k_1^2k_2^2 \tag{25}$$

where $T^{(0)}_{\mu\nu}$ is given by (14). So now at $k_1^2 = k_2^2 = -m^2$

$$k_{1\mu}T_{\mu\nu} = T_{\mu\nu}k_{2\nu} = 0 \tag{26}$$

and hence from the expression (6) for $\epsilon_{\rm L}$ only the term proportional to the product of lightlike vectors $n_{1\mu}n_{2\nu}$ contributes to the amplitude $T_{\rm LL}$. So just as before

$$T_{\rm LL} = A + {\rm O}(m^2/w_1w_2) \tag{27}$$

but the presence of the $\boldsymbol{\phi}$ field in the Lagrangian now implies that the longitudinal polarization vectors $\epsilon_{\rm L}^{(i)}$ can be replaced by the lightlike vectors $n_i$. This is the function of the $\boldsymbol{\phi}$ field for $m \neq 0$; we expect it to be a subsidiary field which does not change the physics. Yao (1973) shows how this occurs in Abelian theories. But at $m = 0$, it becomes physical, and as a corollary the polarization vectors $\epsilon_{\rm L}^{(i)}$ have no meaning in this limit. The vectors $n_i$, however, are continuous in the limit and do behave like scalar fields for $m = 0$ as has been discussed by McKenzie (1972).

Note, however, that the 'charge' of the scalar field $\phi^\pm$ is not $e$, but $\tfrac{1}{2}e$, from (23). So the conflict between Lorentz invariance and the non-Abelian gauge symmetry has not been resolved by the introduction of the $\boldsymbol{\phi}$ field as the $\boldsymbol{\phi}$ field, treated seriously as a physical particle at $m = 0$, is not invariant under the appropriate isotopic transformation which is

$$\boldsymbol{\phi} \to \boldsymbol{\phi} + e\boldsymbol{\Lambda} \times \boldsymbol{\phi}. \tag{28}$$

We conclude therefore that the scalar fields $\boldsymbol{\phi}$ must be included in Yang–Mills theory or other non-Abelian gauge theories to describe the longitudinal modes in the $m = 0$ limit if the theory is to be Lorentz invariant, but that this is a necessary requirement, not a sufficient one, as the $m = 0$ limit itself breaks the non-Abelian gauge symmetry. We shall deal with this problem of symmetry breaking in another paper (Dombey and Vayonakis 1976), together with the problems of tree-unitarity and renormalizability which have been carefully avoided here.

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